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Analyticity of the *d*-Dimensional Bond Percolation Probability Around $p=1$

Gastão A. Braga,[1] Aldo Proccaci,[1] and Remy Sanchis[1]

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Let $\theta(p)$ be the percolation probability of a d-dimensional bond percolation process on Z^d. We prove that $1-\theta(p)$ can be written as an absolutely convergent series in powers of $(1-p)/p$, provided that $|(1-p)/p|$ is sufficiently small. This implies that $\theta(p)$ is an analytic function of the complex variable p, around $p=1$.

KEY WORDS: Percolation probability; polymer expansion; analytic function.

1. INTRODUCTION

Consider a translation invariant bond percolation process on the d-dimensional cubic lattice. A bond is open with probability p and closed with probability $1-p$. Let $\theta(p)$ be the percolation probability of such a process. It is well known that this percolation model presents a phase transition if $d \geqslant 2$, i.e., $\theta(p)$, as a function of $p \in [0, 1]$, is zero for $0 \leqslant p < p_c$ and strictly positive for $p_c < p \leqslant 1$, where p_c is the critical probability, with $0 < p_c < 1$ for $d \geqslant 2$.

The function $\theta(p)$ is known to be C^∞ in the interval $(p_c, 1]$ (see, e.g., ref. 1 and references therein). It remained as an open problem to know whether or not the percolation probability is an analytic function in the same interval. The general belief is that following conjecture is true:

Conjecture. $\theta(p)$ is an analytic function in $(p_c, 1]$.

It is surprising that there is no proof of this conjecture even for p close to 1 since in this regime one expects that cluster-type expansion methods

[1] Departamento de Matemática, UFMG, Caixa Postal 1621, 30161-970 Belo Horizonte, Minas Gerais, Brazil; e-mail: aldo@mat.ufmg.br

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could be applied. We observe that this in not just an academic matter. For instance, there are examples of disordered systems in statistical mechanics that develop a Griffiths singularity, i.e., systems that have a phase transition point even though their free energy is a C^∞ function. In this paper we prove that

Theorem 1.1. $\theta(p)$ is an analytic function of the complex variable p around $p=1$.

Our proof of the above theorem is based upon a polymer gas representation for $1-\theta(p)$. The polymer expansion is a standard technique in equilibrium statistical mechanics and quantum field theory (see, e.g., refs. 2–4) and its has been applied with success to many different problems.

Although a polymer expansion for the percolation model for $p \approx 0$ is immediate (in this case, lattice animals play the role of polymers), it is not clear how to do it for $p \approx 1$. As a matter of fact, in this regime the "statistical weight" of animals is controlled by the closed edges of the animal's boundary, i.e., a lattice animal has a small statistical weight if it has a large number of such closed edges, and open edges are expected to give a negligible contribution to this weight. If one proceeds analogously as in the small p regime, by throwing out the contribution of the open edges, then one can bound above $1-\theta(p)$ by a series in powers of $|1-p|$. At this point one should conclude convergence if the number of lattice animals passing through a fixed point and with n closed edges in the boundary grows as C^n. Unfortunately this is not true because this number actually grows faster than C^n for any positive constant C.

In this paper we show that there is a suitable way of defining polymers so that the polymer expansion for $1-\theta(p)$ converges uniformly for complex values of p close enough to $p=1$. This will imply that $\theta(p)$ is analytic around $p=1$.

We believe that our results can be extended to all values of p inside the interval $(p_c, 1]$. We also point out that our approach to this problem may shed some light on other very interesting questions. For instance, in order to prove the Ornstein–Zernike behaviour of the finite connectivity function in the supercritical regime, one should first establish its analyticity around $p=1$, and we are working on this issue (see ref. 5 for a proof of the O-Z behaviour in the subcritical regime). The methods of this paper can also be extended, with no extra effort, to finite range percolation and, under some conditions on the decay rate of probabilities, to infinite range percolation.

This paper is divided as follows: in Section 2 we give some definitions and we make some remarks on finite volume probabilities; in Section 3 we define the polymers and we provide a polymer gas representation for the finite volume percolation probability; in Section 4 we give some entropy estimates; in Section 5 we prove the main theorem.

2. PRELIMINARIES

We denote the unit square lattice by $\mathbb{Z}^d$. If $R \subset \mathbb{Z}^d$ is finite we denote by $|R|$ the number of its elements. If $x \in \mathbb{Z}^d$ and $y \in \mathbb{Z}^d$, then $\|x-y\|$ denotes the Euclidean distance between x and y. An unordered pair $b = \{x, y\} \subset \mathbb{Z}^d$ such that $\|x-y\| = 1$ is called a *bond* in $\mathbb{Z}^d$. Two distinct bonds b, b' are said to be connected if $b \cap b' \neq \emptyset$. $B(\mathbb{Z}^d)$ denotes the set of bonds in $\mathbb{Z}^d$. To each bond $b \in B(\mathbb{Z}^d)$ it is assigned a random variable ω_b which takes the value 1 (the bond is *open*) with probability p and 0 (the bond is *closed*) with probability $q = 1-p$, i.e., $d\mu(\omega_b) = p\delta(\omega_b - 1)\, d\omega_b + q\delta(\omega_b)\, d\omega_b$. A *configuration* is a function ω: $B(\mathbb{Z}^d) \to \{0, 1\}$ over the set of bonds, assuming the values 0 or 1. The *configuration space* is the set of all such functions, usually denoted by Ω. A (translation invariant) *bond percolation process* is the product measure $P = \prod_b d\mu(\cdot)$, $b \in B(\mathbb{Z}^d)$, defined on the σ-algebra generated by the cylinder sets of Ω.

An *open cluster* of a configuration $\omega \in \Omega$ is a pairwise connected set of open bonds. We denote by C the open cluster that contains the origin. A fundamental quantity of interest is the *percolation probability* $\theta(p)$, which is defined as the probability that the open cluster containing the origin C is infinite, i.e.,

$$\theta(p) = P(|C| = \infty) \tag{2.1}$$

The percolation probability (2.1) can be rewritten as $\theta(p) = 1 - \theta^c(p)$, where

$$\theta^c(p) = P(|C| < +\infty) \tag{2.2}$$

$\theta^c(p)$ can be seen as the (infinite volume) limit of a sequence of (finite volume) functions as follows: let N be a positive integer and consider the box $\Lambda \equiv [-N, N]^d \cap \mathbb{Z}^d$. Let $B(\Lambda)$ denote the set of all bonds in Λ. Define $\partial\Lambda \equiv \{(x_1,\ldots, x_d) \in \Lambda;\ |x_i| = N, \text{ for some } 1 \leqslant i \leqslant d\}$ and $\partial B(\Lambda) \equiv \{b \in B(\Lambda);\ b \cap \partial\Lambda \neq \emptyset\}$, respectively, as the pointwise and bondwise boundary of Λ. Let $\theta^c_\Lambda(p)$ be the probability that the open cluster through the origin does not reach the boundary of Λ:

$$\theta^c_\Lambda(p) = P(C \cap \partial B(\Lambda) = \emptyset) \tag{2.3}$$

The function $\theta^c_\Lambda(p)$ is related to the percolation probability $\theta(p)$ through the limit (the limit $\Lambda \to \infty$ means that $N \to +\infty$)

$$\lim_{\Lambda \to \infty} \theta^c_\Lambda(p) = \theta^c(p) = 1 - \theta(p), \qquad \text{for all} \quad p \in [0, 1] \tag{2.4}$$

In the next section we will rewrite $\theta^c_\Lambda(p)$ as a series in powers of $(1-p)/p$. To reach our goal, we will rewrite $\theta^c_\Lambda(p)$ as a typical statistical mechanics expected value. Given a configuration $\omega_\Lambda \in \Omega_\Lambda$, where Ω_Λ is the configuration space in Λ, we denote by A_{ω_Λ} the set of all open bonds of ω_Λ and by F_{ω_Λ} the set of all closed bonds of ω_Λ. The probability assigned to a given configuration ω_Λ is given by

$$P(\omega_\Lambda) = p^{|A_{\omega_\Lambda}|}(1-p)^{|F_{\omega_\Lambda}|} \tag{2.5}$$

Of course (2.5) is a genuine probability in the sense that

$$1 = \sum_{\omega_\Lambda \in \Omega_\Lambda} P(\omega_\Lambda) \tag{2.6}$$

Defining the function $\lambda(p)$

$$\lambda(p) = \frac{1-p}{p} \tag{2.7}$$

we rewrite (2.6) as follows:

$$1 = p^{|B(\Lambda)|} \sum_{\omega_\Lambda \in \Omega_\Lambda} [\lambda(p)]^{|F_{\omega_\Lambda}|} \tag{2.8}$$

Based on (2.8), we introduce the function

$$Z_\Lambda(\lambda) = \sum_{\omega_\Lambda \in \Omega_\Lambda} \lambda^{|F_{\omega_\Lambda}|} \tag{2.9}$$

where $\lambda \in \mathbb{C}$, and we rewrite (2.8) as

$$p^{|B(\Lambda)|} = \frac{1}{Z_\Lambda(\lambda(p))} \tag{2.10}$$

Explicitly, we have a representation for $\theta^c_\Lambda(p)$ as

$$\theta^c_\Lambda(p) = \sum_{\substack{\omega_\Lambda \in \Omega_\Lambda \\ C \cap \partial\Lambda = \varnothing}} p^{|A_{\omega_\Lambda}|}(1-p)^{|F_{\omega_\Lambda}|} = \frac{1}{Z_\Lambda(\lambda(p))} \sum_{\substack{\omega_\Lambda \in \Omega_\Lambda \\ C \cap \partial\Lambda = \varnothing}} [\lambda(p)]^{|F_{\omega_\Lambda}|} \tag{2.11}$$

The ratio given on the r.h.s. of (2.11) allows us to reinterpret $\theta^c_\Lambda(p)$ as an expected value in sense of classical statistical mechanics. Starting from this ratio, we develop a representation for $\theta^c_\Lambda(p)$ in terms of suitable geometrical objects, called polymers, which will allow us to use the polymer expansion in order to prove Theorem 1.1.

3. THE POLYMER EXPANSION

We are interested in studying the analytical properties of the percolation probability $\theta(p)$ in the region near $p=1$. To do so, we will initially show that the function $\theta^c_\Lambda(p)$ can be represented, near $p=1$, as a ratio involving a suitable partition function. Given a configuration $\omega_\Lambda \in \Omega_\Lambda$, we associate to each closed bond $b \in F_{\omega_L}$ a $(d-1)$-dimensional unit hypersquare σ which cuts perpendicularly the closed bond in the middle point. The vertices of the hypersquare lay in the so called dual lattice Λ^*. Two hypersquares σ and σ' are said to be connected if they share a $(d-2)$-dimensional side.

We define a *dual lattice animal* γ as the (pairwise) connected union of such hypersquares. We say that two dual animals γ_i and γ_j are *compatible*, and we write it as $\gamma_i \sim \gamma_j$, if $\dim(\gamma_i \cap \gamma_j) \leqslant d-3$. We denote by Γ_Λ the set of dual animals in Λ^*, while Γ will denote the set of dual animals in $\mathbb{Z}^{d^*}$. A configuration ω_Λ determines uniquely a configuration of (pairwise) compatible dual animals $\{\gamma_1,..., \gamma_n\} \subset \Gamma_\Lambda$ on the dual lattice. We will sometimes regard a dual animal as a $(d-1)$-dimensional surface in $\mathbb{R}^d$. The *interior* of a dual animal is the union of the bounded connected components of $\mathbb{R}^d - \gamma$ and it will be denoted by $I(\gamma)$ (see Fig. 1 for a two-dimensional example).

We will denote by $|\gamma|$ the number of hyper-squares which form γ. Finally, to any dual animal γ we associate a a statistical weight $\lambda^{|\gamma|}$.

With these notations the function (2.9) can be rewritten as

$$Z_\Lambda(\lambda) = 1 + \sum_{n \geqslant 1} \sum_{\substack{\{\gamma_1,..., \gamma_n\} \subset \Gamma_\Lambda \\ \gamma_i \sim \gamma_j}} \lambda^{|\gamma_1|} \cdots \lambda^{|\gamma_n|} \tag{3.1}$$

where the factor 1 corresponds to the configuration ω_Λ where all bonds are open (hence no dual animals are present). The r.h.s. of (3.1) is the standard grand canonical partition function of a hard core gas of compatible dual animals γ with activities $\lambda^{|\gamma|}$ (see, e.g., refs. 3 and 6)

The function $\theta^c_\Lambda(p)$ can also be expressed in terms of dual animals. In fact, note that the constraint in the summation of r.h.s. of (2.11) to sum

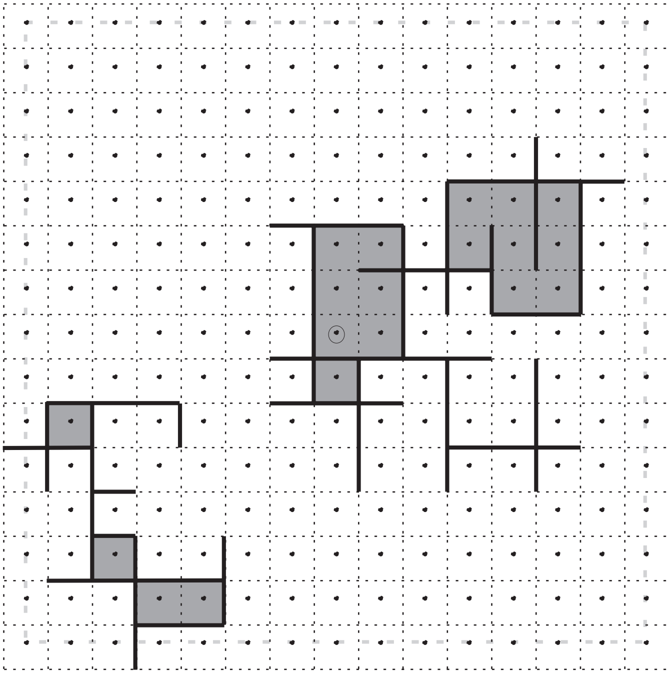

Fig. 1. Two compatible dual animals in Λ^*. Their interiors are the regions marked in grey. The dual animal on the right does not reach the boundary $\partial B(\Lambda)$ and has the origin is its interior. The dual animal on the left reaches $\partial B(\Lambda)$.

over those ω_Λ such that $C \cap \partial B(\Lambda) = \emptyset$ can be rephrased as to sum over dual animal configurations $\{\gamma_1,..., \gamma_n\}$ such that there exists γ_i with $0 \in I(\gamma_i)$. Thus, recalling (2.11) we get

$$\theta^c_\Lambda(p) = \frac{\sum_{n\geqslant 1} \sum_{\substack{\{\gamma_1,\dots,\gamma_n\} \subset \Gamma_\Lambda : \gamma_i \sim \gamma_j \\ \exists \gamma_k : 0 \in I(\gamma_k)}} \lambda^{|\gamma_1|} \dots \lambda^{|\gamma_n|}}{1 + \sum_{n\geqslant 1} \sum_{\substack{\{\gamma_1,\dots,\gamma_n\} \subset \Gamma_\Lambda \\ \gamma_i \sim \gamma_j}} \lambda^{|\gamma_1|} \dots \lambda^{|\gamma_n|}} \tag{3.2}$$

We will now re-express $\theta^c_\Lambda(p)$ as a series in powers of λ. In order to do that we will need some new definitions.

Definition 3.1. A *polymer* $\tilde{\gamma}$ will be a collection of dual animals $\{\gamma_1,..., \gamma_n\}$ satisfying the following conditions:

- $n = 1$ or
- $\gamma_i \sim \gamma_j$ and $0 \in I(\gamma_i)$ for all $i, j = 1, 2...n$, with $n \geqslant 2$.

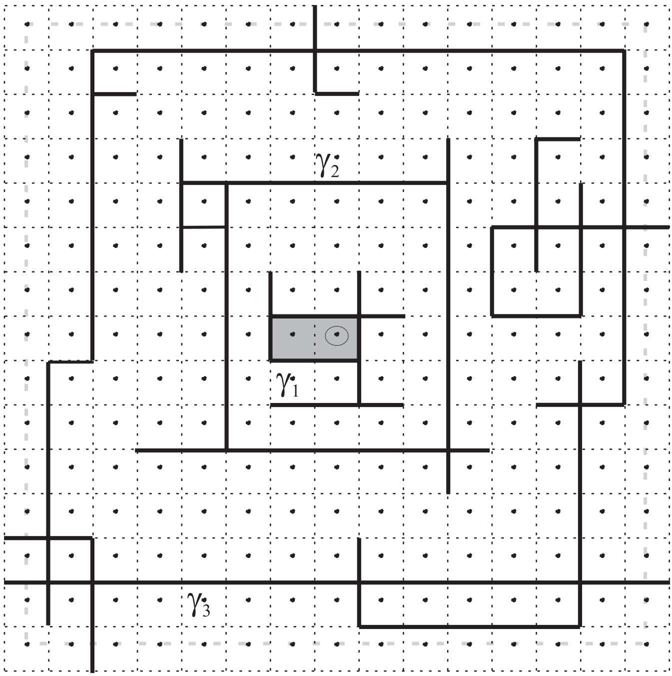

Fig. 2. A polymer $\tilde{\gamma}$ made by three dual animals γ_1, γ_2 and γ_3. The area in grey is the interior of $\tilde{\gamma}$.

See Fig. 2 for a two dimensional example of a polymer. The set of all polymers contained in Λ^* will be denoted by $\tilde{\Gamma}_\Lambda$, while the set of all polymers contained in $\mathbb{Z}^{d^*}$ will be denoted by $\tilde{\Gamma}$. Defining the interior of $\tilde{\gamma}$ as $I(\tilde{\gamma}) \equiv \bigcap_{i=1}^n I(\gamma_i)$, we have a new notion of compatibility:

Definition 3.2. We say that the polymers $\tilde{\gamma}_\alpha$ and $\tilde{\gamma}_\beta$ are compatible if

- $\gamma_a \sim \gamma_b$ for all $\gamma_a \in \tilde{\gamma}_\alpha$ and all $\gamma_b \in \tilde{\gamma}_\beta$.
- $0 \notin I(\tilde{\gamma}_\alpha)$ or $0 \notin I(\tilde{\gamma}_\beta)$.

We denote the compatibility of $\tilde{\gamma}_\alpha$ and $\tilde{\gamma}_\beta$ by $\tilde{\gamma}_\alpha \approx \tilde{\gamma}_\beta$.

We denote by $|\tilde{\gamma}|$ the number of hyper-squares in $\tilde{\gamma}$, i.e., $|\tilde{\gamma}| = \sum_i |\gamma_i|$.

The statistical weight of the polymer $\tilde{\gamma}$ is $\lambda^{|\tilde{\gamma}|}$. By construction we have that

$$1 + \sum_{n \geqslant 1} \sum_{\substack{\{\gamma_1, \dots, \gamma_n\} \subset \Gamma_\Lambda \\ \gamma_i \sim \gamma_j}} \lambda^{|\gamma_1|} \cdots \lambda^{|\gamma_n|} = 1 + \sum_{n \geqslant 1} \sum_{\substack{\{\tilde{\gamma}_1, \dots, \tilde{\gamma}_n\} \subset \tilde{\Gamma}_\Lambda \\ \tilde{\gamma}_i \approx \tilde{\gamma}_j}} \lambda^{|\tilde{\gamma}_1|} \cdots \lambda^{|\tilde{\gamma}_n|} \tag{3.3}$$

and that

$$\sum_{n\geqslant 1}\sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\subset\Gamma_\Lambda\\ \gamma_i\sim\gamma_j,\ \exists\gamma_k:0\in I(\gamma_k)}}\lambda^{|\gamma_1|}\cdots\lambda^{|\gamma_n|}=\sum_{n\geqslant 1}\sum_{\substack{\{\tilde\gamma_1,\dots,\tilde\gamma_n\}\subset\tilde\Gamma_\Lambda\\ \tilde\gamma_i\approx\tilde\gamma_j,\ \exists!\tilde\gamma_k:0\in I(\tilde\gamma_k)}}\lambda^{|\tilde\gamma_1|}\cdots\lambda^{|\tilde\gamma_n|}\tag{3.4}$$

Note that in the sum over the polymer configurations $\{\tilde\gamma_1,\dots,\tilde\gamma_n\}$ on the r.h.s. of (3.4) just *one and only one* polymer contains the origin in its interior. This is a crucial difference with the sum in l.h.s of (3.4).

By (3.3) and (3.4) we can rewrite

$$\theta^c_\Lambda(p)=\frac{\displaystyle\sum_{n\geqslant 1}\sum_{\substack{\{\tilde\gamma_1,\dots,\tilde\gamma_n\}\subset\tilde\Gamma_\Lambda\\ \tilde\gamma_i\approx\tilde\gamma_j,\ \exists!\tilde\gamma_k:0\in I(\tilde\gamma_k)}}\lambda^{|\tilde\gamma_1|}\cdots\lambda^{|\tilde\gamma_n|}}{\displaystyle 1+\sum_{n\geqslant 1}\sum_{\substack{\{\tilde\gamma_1,\dots,\tilde\gamma_n\}\subset\tilde\Gamma_\Lambda\\ \tilde\gamma_i\approx\tilde\gamma_j}}\lambda^{|\tilde\gamma_1|}\cdots\lambda^{|\tilde\gamma_n|}}\tag{3.5}$$

where λ is evaluated at $\lambda(p)$.

Now the polymer representation (3.5) allows us to express the function $\theta^c_\Lambda(p)$ as a derivative of the logarithm of a suitable partition function. To see it, we define a new activity for the polymers $\tilde\gamma$. Let $\alpha\in\mathbb{R}$ and $\tilde\gamma\in\tilde\Gamma$ and define

$$\rho_\alpha(\tilde\gamma)=\begin{cases}(1+\alpha)\,\lambda^{|\tilde\gamma|} & \text{if}\quad 0\in I(\tilde\gamma)\\ \lambda^{|\tilde\gamma|} & \text{otherwise}\end{cases}\tag{3.6}$$

Let $\Xi_{\Lambda,\alpha}(\lambda)$ denote the grand canonical partition function

$$\Xi_{\Lambda,\alpha}(\lambda)=1+\sum_{n\geqslant 1}\sum_{\substack{\{\tilde\gamma_1,\dots,\tilde\gamma_n\}\subset\tilde\Gamma_\Lambda\\ \tilde\gamma_i\approx\tilde\gamma_j}}\rho_\alpha(\tilde\gamma_1)\cdots\rho_\alpha(\tilde\gamma_n)\tag{3.7}$$

and let

$$f_\Lambda(\lambda)=\frac{\partial}{\partial\alpha}\bigg|_{\alpha=0}\ln\Xi_{\Lambda,\alpha}(\lambda)\tag{3.8}$$

Then, one can now easily check that

$$\theta^c_\Lambda(p)=f_\Lambda(\lambda(p))\tag{3.9}$$

The great advantage of formulas (3.7)–(3.9) is that they allow us to re-express $f_\Lambda(\lambda)$, and consequently $\theta^c_\Lambda(p)$, directly as a series instead of a ratio between two (finite) sums as it is in (3.5). As a matter of fact, the r.h.s. of (3.7) is again the grand canonical partition function of a hard core polymer gas in which the polymers are objects in $\tilde\Gamma_\Lambda$, with activities given by $\rho_\alpha(\tilde\gamma)$

and interacting via a hard core potential (in the sense that they must be compatible). Namely, defining

$$U(\tilde{\gamma}_i, \tilde{\gamma}_j) = \begin{cases} +\infty & \text{if} \quad \tilde{\gamma}_i \not\sim \tilde{\gamma}_j \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

the r.h.s. of (3.7) can be rewritten as

$$\Xi_{\Lambda,\alpha}(\lambda) = 1 + \sum_{n \geqslant 1} \frac{1}{n!} \sum_{(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n) \in (\tilde{\Gamma}_\Lambda)^n} \rho_\alpha(\tilde{\gamma}_1) \cdots \rho_\alpha(\tilde{\gamma}_n)\, e^{-\sum_{1 \leqslant i<j \leqslant n} U(\tilde{\gamma}_i, \tilde{\gamma}_j)} \tag{3.11}$$

where $(\tilde{\Gamma}_\Lambda)^n$ is the n-times cartesian product of $\tilde{\Gamma}_\Lambda$.

It is thus a standard task in cluster expansion theory to compute explicitly the Mayer expansion for the function $\ln \Xi_{\Lambda,\alpha}(\lambda)$ (see, e.g., refs. 4 and 6). Doing such a calculation one obtains

$$\ln \Xi_{\Lambda,\alpha}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\tilde{\gamma}_1\ldots\tilde{\gamma}_n) \in (\tilde{\Gamma}_\Lambda)^n} \Phi^T(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n)\, \rho_\alpha(\tilde{\gamma}_1)\ldots\rho_\alpha(\tilde{\gamma}_n) \tag{3.12}$$

where the Ursell coefficients $\Phi^T(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n)$ are given by

$$\Phi^T(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n) = \begin{cases} \sum_{g \in G_n} \prod_{\{i,j\} \in g} (e^{-U(\tilde{\gamma}_i, \tilde{\gamma}_j)} - 1) & \text{if} \quad n \geqslant 2 \\ 1 & \text{if} \quad n = 1 \end{cases} \tag{3.13}$$

The sum in (3.13) is over all connected graphs on $\{1, 2,\ldots, n\}$. It is important to stress that the sum in the r.h.s. of (3.12) is actually an infinite series, while the sums in (3.5) are finite. In the next section we will show that such a series is absolutely convergent for $|\lambda|$ sufficiently small uniformily in Λ. Deriving it term by term with respect to α and evaluating the result at $\alpha = 0$, it is clear, by definition (3.6), that the only non vanishing terms are those associated to configurations $\tilde{\gamma}_1,\ldots, \tilde{\gamma}_n$ in which at least one among the $\tilde{\gamma}_i$'s contains the origin in its interior. Thus we obtain

$$f_\Lambda(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\tilde{\gamma}_1\cdots\tilde{\gamma}_n) \in (\tilde{\Gamma}_\Lambda)^n \\ \exists! \tilde{\gamma}_k : 0 \in I(\tilde{\gamma}_k)}} k(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n)\, \Phi^T(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n)\, \lambda^{\sum_{i=1}^n |\tilde{\gamma}_i|} \tag{3.14}$$

The positive integer $k(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n)$, is the number of polymers in $(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n)$ having the origin in its interior. Note that $k(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_n) \leqslant n$. We also remark that r.h.s. of (3.14) is a series in powers of λ. In the sequel, we will show that

Theorem 3.1. There exists a positive constant λ_0, which is independent of the volume Λ, such that the series given by Eq. (3.14) converges absolutely for complex values of λ inside the disk $|\lambda| < \lambda_0$.

Remark. From the proof of the above theorem, we will see that $\lambda_0 = 1/(6 \cdot 2^d e C_0)$, where C_0 is given in Lemma 4.1.

A standard corollary of the above theorem (see, e.g., ref. 3, Theorem 20.4.2) is

Corollary 3.1. For any box Λ, the function $f_\Lambda(\lambda)$ in (3.14) is analytic in the complex disk $|\lambda| < \lambda_0$ and the limit

$$f(\lambda) = \lim_{\Lambda \to \infty} f_\Lambda(\lambda) \tag{3.15}$$

exists and is analytic in the same disk.

We postpone the proof of Theorem 3.1 and Corollary 3.1 to Section 5.

Remark. By Corollary 3.1, the function $f((1-z)/z)$ is analytic in the complex domain $D = \{z \in \mathbb{C} : |(1-z)/z| < \lambda_0\}$ and, by (2.4) and (3.9), $f((1-p)/p) = \theta^c(p)$ for $p \in [0, 1]$. Hence $f((1-z)/z)$ is the unique analytic continuation of $\theta^c(p)$ to the domain D. This proves Theorem 1.1.

4. ENTROPY ESTIMATES

In order to prove Theorem 3.1 we will need some preliminary lemmas related to entropy estimates.

Let us thus denote by $\mathbb{S}^d$ the set of all $(d-1)$-dimensional hyper-squares σ on the dual lattice $\mathbb{Z}^{d^*}$. Given $\sigma \in \mathbb{S}^d$, let us denote by S_d the number of hyper-squares σ' incompatible with σ and not equal to σ. Note that $S_d = 6(d-1)$. We will initially prove that

Lemma 4.1. The number of dual animals γ of a given size $|\gamma| = n$ that pass through a given point $x^* \in \mathbb{Z}^{d^*}$ is at most exponential in $|\gamma|$, i.e., for all $x^* \in \mathbb{Z}^{d^*}$ and $n \in \mathbb{N}$

$$\sum_{\substack{\gamma : x^* \in \gamma \\ |\gamma| = n}} 1 \leqslant C_0^n \tag{4.1}$$

where $C_0 = S_d^{-1}(1 + S_d)^{1+S_d}$

Proof. Let 0^* be the origin of the dual lattice. By translation invariance, it is enough to show that

$$\sum_{\substack{\gamma : 0^* \in \gamma \\ |\gamma| = n}} 1 \leqslant C_0^n \tag{4.2}$$

We start observing that the probability to have a dual animal γ is given by $P(\gamma) = (1-p)^{|\gamma|} p^{|\partial\gamma|}$ where $\partial\gamma$ is the boundary of the dual animal γ defined as $\partial\gamma = \{\sigma \notin \gamma : \sigma \not\sim \sigma', \text{ for some } \sigma' \in \gamma\}$. Noting that $|\partial\gamma| \leqslant S_d\,|\gamma|$, we have

$$\sum_{\substack{\gamma : 0^* \in \gamma \\ |\gamma| = n}} (1-p)^{|\gamma|}\, p^{S_d\,|\gamma|} \leqslant \sum_{\substack{\gamma : 0^* \in \gamma \\ |\gamma| = n}} (1-p)^{|\gamma|}\, p^{|\partial\gamma|} < 1$$

where the middle term in the above inequality is the probability of having a dual animal of size n passing by 0^*. Thus we have

$$\sum_{\substack{\gamma : 0^* \in \gamma \\ |\gamma| = n}} 1 \leqslant \left[\frac{1}{(1-p)\, p^{S_d}}\right]^n \leqslant \left\{(1+S_d)\left[\frac{1+S_d}{S_d}\right]^{S_d}\right\}^n$$

and therefore the lemma is proved. ∎

We next prove the following lemma.

Lemma 4.2. The number of dual animals γ of a given cardinality $|\gamma| = n$ such that $0 \in I(\gamma)$ is at most exponential in n, i.e., for all $n \in \mathbb{N}$

$$\sum_{\substack{\gamma : 0 \in I(\gamma) \\ |\gamma| = n}} 1 \leqslant C_1^n \tag{4.3}$$

where $C_1 = 2^d C_0$ and C_0 is given in Lemma 4.1.

Proof. Using that the volume of the interior of a dual animal γ of cardinality n is at most n^d and applying Lemma 4.1, we get

$$\sum_{\substack{\gamma : I(\gamma) \neq \varnothing \\ |\gamma| = n}} 1 \leqslant n^d \sum_{\substack{\gamma : 0^* \in \gamma \\ |\gamma| = n}} 1 \leqslant 2^{dn} C_0^n \leqslant C_1^n \quad ∎$$

Finally, we prove the main entropy estimate

Lemma 4.3. The number of polymers $\tilde{\gamma}$ of a given cardinality $|\tilde{\gamma}| = n$ and with $0 \in I(\tilde{\gamma})$ is at most exponential in n, i.e., for all $n \in \mathbb{N}$

$$\sum_{\substack{\tilde{\gamma} : 0 \in I(\tilde{\gamma}) \\ |\tilde{\gamma}| = n}} 1 \leqslant C_2^n \tag{4.4}$$

where $C_2 = 2C_1$ and C_1 is given in Lemma 4.2.

Proof. Recall that a polymer $\tilde{\gamma}$ is a collection of compatible dual animals the form $\tilde{\gamma} = \{\gamma_1,..., \gamma_k\}$ such that $I(\gamma_i) \ni 0$. Hence if $|\tilde{\gamma}| = n$, then clearly $1 \leqslant k \leqslant n$. Using thus Lemma 4.2, it is straightfoward to see that the sum (4.4) is bounded above by

$$\begin{aligned} &\sum_{k=1}^{n} \sum_{|\gamma_1| + \cdots + |\gamma_k| = n} \prod_{i=1}^{k} \Bigg(\sum_{\substack{\gamma : 0 \in I(\gamma) \\ |\gamma| = |\gamma_i|}} 1 \Bigg) \\ &\leqslant \sum_{k=1}^{n} \sum_{|\gamma_1| + \cdots + |\gamma_k| = n} \prod_{i=1}^{k} (C_1^{|\gamma_i|}) = \sum_{k=1}^{n} \sum_{|\gamma_1| + \cdots + |\gamma_k| = n} C_1^n \\ &\leqslant C_1^n \sum_{k=1}^{n} \sum_{|\gamma_1| + \cdots + |\gamma_k| = n} 1 \leqslant C_1^n \sum_{k=1}^{n} \binom{n-1}{k-1} = C_1^n 2^{n-1} \leqslant C_2^n \quad \blacksquare \end{aligned} \tag{4.5}$$

5. PROOF OF THEOREM 3.1

We will bound, uniformly in Λ, the series given by Eq. (3.14). We remind that the polymer configurations $\{\tilde{\gamma}_1,..., \tilde{\gamma}_n\}$ contributing to the sum (3.14) have at least one polymer whose interior contains the origin. Thus we get

$$|f_\Lambda(\lambda)| \leqslant \sum_{\tilde{\gamma}_1 \in \tilde{\Gamma}_\Lambda : I(\tilde{\gamma}_1) \ni 0} |\lambda|^{|\tilde{\gamma}_1|} \left[1 + \sum_{n \geqslant 2} \frac{n}{(n-1)!} B_{n,\Lambda}(\tilde{\gamma}_1) \right] \tag{5.1}$$

where

$$B_{n,\Lambda}(\tilde{\gamma}) \equiv \sum_{(\tilde{\gamma}_2 \ldots \tilde{\gamma}_n) \in (\tilde{\Gamma}_\Lambda)^{n-1}} |\Phi^T(\tilde{\gamma}, \tilde{\gamma}_2,..., \tilde{\gamma}_n) \, \lambda^{\sum_{i=2}^{n} |\tilde{\gamma}_i|}|$$

To continue, we need an upper bound for $B_{n,\Lambda}(\tilde{\gamma})$. Define

$$v \equiv \sum_{\substack{\tilde{\gamma} \in \tilde{\Gamma} \\ 0^* \in \tilde{\gamma}}} (e\,|\lambda|)^{|\tilde{\gamma}|} + \sum_{\substack{\tilde{\gamma} \in \tilde{\Gamma} \\ 0 \in I(\tilde{\gamma})}} (e\,|\lambda|)^{|\tilde{\gamma}|} \tag{5.2}$$

Note that v does not depend on Λ for in (5.2) we are summing on $\tilde{\Gamma}$ rather than on $\tilde{\Gamma}_\Lambda$. In the sequel we will make use of the following lemma whose proof appears below.

Lemma 5.1. For $n \geqslant 2$, for any Λ and for any polymer $\tilde{\gamma}$:

$$B_{n,\Lambda}(\tilde{\gamma}) \leqslant (n-1)!\, e^{|\tilde{\gamma}|}[2^d v]^{n-1} \tag{5.3}$$

Plugging estimate (5.3) into (5.1), we get

$$|f_\Lambda(\lambda)| \leqslant \sum_{\tilde{\gamma} \in \tilde{\Gamma}_\Lambda : I(\tilde{\gamma}) \ni 0} |\lambda|^{|\tilde{\gamma}|} \left[1 + e^{|\tilde{\gamma}|} \sum_{n \geqslant 2} n(2^d v)^{n-1} \right]. \tag{5.4}$$

The series $\sum_{n \geqslant 2} n(2^d v)^{n-1}$ will converge and will be bounded by 3 if $2^d v < 1/2$ and this condition will be achieved if $|\lambda| \leqslant 1/(4^d 12 e C_0)$, where C_0 is the constant in Lemma 4.1. As a matter of fact, using Lemmas 4.2 and 4.3, we get

$$v \leqslant \sum_{n \geqslant 1} \left[\sum_{\substack{\tilde{\gamma} : 0^* \in \tilde{\gamma} \\ |\tilde{\gamma}| = n}} (|\lambda|\, e)^n + \sum_{\substack{\tilde{\gamma} : 0 \in I(\tilde{\gamma}) \\ |\tilde{\gamma}| = n}} (|\lambda|\, e)^n \right] \leqslant \sum_{n \geqslant 1} [(|\lambda|\, eC_1)^n + (|\lambda|\, eC_2)^n]$$

Remembering that $C_2 = 2C_1$, the last sum above converges if $|\lambda|\, eC_1 < 1/2$ and it is less than $6|\lambda|\, eC_1$ if $|\lambda|\, eC_1 < 1/4$. Thus, recalling that $C_1 = 2^d C_0$ where C_0 is the constant of Lemma 4.1, the condition $2^d v < 1/2$ is achieved if $|\lambda| < 1/(4^d 12 e C_0)$. Using now the upper bound (5.4), using Lemma 4.3 once again and using that $\sum_{n \geqslant 2} n(2^d v)^{n-1} < 3$ for λ inside the disc $|\lambda| < 1/(4^d 12 e C_0)$, we get

$$\begin{aligned} |f_\Lambda(\lambda)| &\leqslant \sum_{\substack{\tilde{\gamma} \in \tilde{\Gamma}_\Lambda \\ I(\tilde{\gamma}) \ni 0}} |\lambda|^{|\tilde{\gamma}|}\, [1 + 3e^{|\tilde{\gamma}|}] \leqslant \sum_{\substack{\tilde{\gamma} \in \tilde{\Gamma} \\ I(\tilde{\gamma}) \ni 0}} |\lambda|^{|\tilde{\gamma}|}\, [1 + 3e^{|\tilde{\gamma}|}] \\ &\leqslant \sum_{n \geqslant 2d} \sum_{\substack{\tilde{\gamma} : 0 \in I(\tilde{\gamma}) \\ |\tilde{\gamma}| = n}} [|\lambda|^{|\tilde{\gamma}|} + 3(e\,|\lambda|)^{|\tilde{\gamma}|}] \\ &\leqslant \sum_{n \geqslant 2d} [(C_2\,|\lambda|)^n + 3(C_2\,|\lambda|\, e)^n] \leqslant \sum_{n \geqslant 2d} [(2^d 2 C_0\,|\lambda|)^n + 3(2^d 2 C_0\,|\lambda|\, e)^n] \end{aligned}$$

The last sum is absolutely convergent for $|\lambda| < 1/(4^d 12 e C_0)$ and it is bounded at least by

$$|f_\Lambda(\lambda)| \leqslant 8(E_d\,|\lambda|)^{2d} \qquad \text{if} \quad |\lambda| < \frac{1}{4^d 12 e C_0}$$

where $E_d \leqslant (2^{d+1}eC_0)$. Note that $E_d\,|\lambda| \leqslant 1/6 \cdot 2^d < 1/6$ in the whole disc $|\lambda| < 1/4^d 12eC_0$. We stress that such bounds are far from optimal. ∎

Proof of Lemma 5.1. To get the upper bound (5.3), we will make use of the following well known bound

$$\left| \sum_{g \in G_n} \prod_{\{i,j\} \in g} (e^{-U(\tilde{\gamma}_i, \tilde{\gamma}_j)} - 1) \right| \leqslant \sum_{\tau \in T_n} \prod_{\{i,j\} \in \tau} |e^{-U(\tilde{\gamma}_i, \tilde{\gamma}_j)} - 1| \tag{5.5}$$

where the sum on the left is over all connected graphs on $\{1, 2,..., n\}$ and the sum on the right is over the tree graphs on $\{1, 2,..., n\}$ (see, e.g., ref. 2).

For a fixed $\tau \in T_n$, it is now easy to calculate exactly the factor $\prod_{\{i,j\} \in \tau} |e^{-\tilde{U}(\tilde{\gamma}_i, \tilde{\gamma}_j)} - 1|$. As a matter of fact, let $g(\tilde{\gamma}_1,..., \tilde{\gamma}_n)$ bet the graph on $\{1, 2,..., n\}$ (not necessarily connected) defined by

$$\{i,j\} \in g(\tilde{\gamma}_1,...,\tilde{\gamma}_n) \qquad \text{if and only if} \quad \tilde{\gamma}_i \not\sim \tilde{\gamma}_j \tag{5.6}$$

Hence, recalling definition (3.10), we have that

$$\prod_{\{i,j\} \in \tau} |e^{-\tilde{U}(\tilde{\gamma}_i, \tilde{\gamma}_j)} - 1| = \begin{cases} 1 & \text{if} \quad \tau \subset g(\tilde{\gamma}_1,..., \tilde{\gamma}_n) \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

where $\tau \subset g(\tilde{\gamma}_1,..., \tilde{\gamma}_n)$ means that if $\{i, j\} \in \tau$ then $\{i, j\} \in g(\tilde{\gamma}_1,..., \tilde{\gamma}_n)$. Thus, using (5.5) and (5.7) we can bound $|B_{n,\Lambda}(\tilde{\gamma}_1)|$ from above as

$$\begin{aligned} B_{n,\Lambda}(\tilde{\gamma}_1) &\leqslant \sum_{\tau \in T_n} \left[\sum_{(\tilde{\gamma}_2 \cdots \tilde{\gamma}_n) \in (\tilde{\Gamma})^{n-1}} \prod_{\{i,j\} \in \tau} |e^{-U(\tilde{\gamma}_i,\tilde{\gamma}_j)} - 1|\ \lambda^{\sum_{i=2}^n |\tilde{\gamma}_i|} \right] \\ &\leqslant \sum_{\tau \in T_n} \left[\sum_{\substack{(\tilde{\gamma}_2 ... \tilde{\gamma}_n) \in (\Gamma_\Lambda)^{n-1} \\ g(\tilde{\gamma}_1,...,\tilde{\gamma}_n) \supset \tau}} |\lambda|^{|\tilde{\gamma}_2|} \cdots |\lambda|^{|\tilde{\gamma}_n|} \right] \equiv \sum_{\tau \in T_n} \omega(\tau) \end{aligned}$$

To obtain an upper bound for $\omega(\tau)$, we will use the following inequality. Let $\tilde{\gamma}_0$ be fixed and let $F(\tilde{\gamma})$ be a given positive function. Then

$$\begin{aligned} \sum_{\tilde{\gamma} : \tilde{\gamma} \not\sim \tilde{\gamma}_0} F(\tilde{\gamma}) &\leqslant 2^d\, |\tilde{\gamma}_0| \sup_{x^* \in \tilde{\gamma}_0} \sum_{\tilde{\gamma} : x^* \in \tilde{\gamma}} F(\tilde{\gamma}) + \sum_{\tilde{\gamma} : 0 \in I(\tilde{\gamma})} F(\tilde{\gamma}) \\ &\leqslant 2^d\, |\tilde{\gamma}_0| \left[\sum_{\tilde{\gamma} : 0^* \in \tilde{\gamma}} F(\tilde{\gamma}) + \sum_{\tilde{\gamma} : 0 \in I(\tilde{\gamma})} F(\tilde{\gamma}) \right] \\ &\equiv 2^d\, |\tilde{\gamma}_0| \ \sum_{\tilde{\gamma}}{}^{*}\ F(\tilde{\gamma}) \end{aligned} \tag{5.8}$$

Fixed the tree graph τ, let d_i be the degree of the vertex i. Evaluating the sum $\omega(\tau)$ by summing from the outermost polymers in the graph $g(\tilde{\gamma}_1,..., \tilde{\gamma}_n) \supset \tau$ and using (5.8), we get

$$\omega(\tau) \leqslant [2^d \, |\tilde{\gamma}_1|]^{d_1} \prod_{i=2}^{n} \left[\sum_{\tilde{\gamma}}^{*} \, [2^d \, |\tilde{\gamma}_i|]^{d_i - 1} \, |\lambda|^{|\tilde{\gamma}_i|} \right]$$

Therefore, using the above estimate and the Cayley's formula (A.1) again, we get

$$\begin{aligned} B_{n,\Lambda}(\tilde{\gamma}_1) &\leqslant \sum_{\tau \in T_n} \omega(\tau) \\ &\leqslant \sum_{d_1 + \cdots + d_n = 2n-2} [2^d \, |\tilde{\gamma}_1|]^{d_1} \frac{(n-2)!}{\prod_{i=1}^{n} (d_i - 1)!} \prod_{i=2}^{n} \left[\sum_{\tilde{\gamma}}^{*} \, [2^d \, |\tilde{\gamma}_i|]^{d_i\text{-}1} \, |\lambda|^{|\tilde{\gamma}_i|} \right] \\ &\leqslant 2^{(n-1)\, d} \, (n-1)! \sum_{d_1 = 1}^{\infty} \frac{|\tilde{\gamma}_1|^{d_1}}{d_1!} \prod_{i=2}^{n} \left[\sum_{\tilde{\gamma}}^{*} \sum_{d_i = 1}^{\infty} \frac{|\tilde{\gamma}_i|^{d_i - 1}}{(d_i - 1)!} \, |\lambda|^{|\tilde{\gamma}_i|} \right] \\ &\leqslant 2^{(n-1)\, d} (n-1)! \, e^{|\tilde{\gamma}_1|} \prod_{i=2}^{n} \left[\sum_{\tilde{\gamma}}^{*} \, (e \, |\lambda|)^{|\tilde{\gamma}_i|} \right] \leqslant (n-1)! \, e^{|\tilde{\gamma}|} [2^d v]^{n-1} \end{aligned}$$

Sketch of the Proof of Corollary 3.1. Concerning the proof of Corollary 3.1, it is obvious from the proof of Theorem 3.1 that $f_\Lambda(\lambda)$ is analytic in λ in the disc $|\lambda| < \lambda_0$ uniformly in Λ. The existence and analyticity of the limit $f(\lambda) = \lim_{\Lambda \to \infty} f_\Lambda(\lambda)$ follows by proving that $f_\Lambda(\lambda)$ is, as $N \to \infty$ (and hence $\Lambda \to \infty$), a Cauchy sequence uniformly in the disc $|\lambda| < \lambda_0$. This is a text book exercise in the theory of cluster expansion (see ref. 3, Theorem 20.4.2). One has just to observe that $f_\Lambda(\lambda) - f_{\Lambda'}(\lambda)$ (supposing $\Lambda \subset \Lambda'$) can be written in term of a sum over n-ple of polymers $(\tilde{\gamma}_1,..., \tilde{\gamma}_n)$ in which all polymers have non void intersection with $\Lambda' \setminus \Lambda$ and at least one of them has interior containing the origin. Thus the power series in λ of $f_\Lambda(\lambda) - f_{\Lambda'}(\lambda)$ starts at least with the power $|\lambda|^{d(0, \partial\Lambda)}$ where $d(0, \partial\Lambda)$ is the minimum distance between the origin and the boundary of Λ. This power takes $f_\Lambda(\lambda) - f_{\Lambda'}(\lambda)$ to zero as $\Lambda \to \infty$. ∎

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